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HIGHER ORDER EFFECTS IN THICK RECTANGULAR ELASTIC BEAMS*

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Abstract—Governing equations for the rectangular strip are investigated to obtain solutions which show the relationship between classical beam theory and higher order theories. The exact equations of plane elasticity are reduced to a coupled set of ordinary differential equations by expressing all dependent variables as series solutions containing Legendre Polynomials in the thickness coordinate. Legendre Polynomials are particularly advantageous for this analysis because their completeness, convergence and orthogonality properties are well formulated, and because the usual stress resultants of classical beam theory appear naturally as coefficients of the polynomials P_0 and P_1. The coupled ordinary differential equations are obtained in a form such that proper truncation of the series to obtain approximate theories is immediately apparent. The coupling effects are investigated and a possible method of obtaining an approximate solution to the fully coupled equations without truncation of the series solutions is suggested. A sample problem is worked out in detail to illustrate the application of a new approximate theory.

NOTATION

L	length of rectangular strip
h	thickness of rectangular strip
x, y	axial, thickness coordinates
η	dimensionless thickness coordinate $= 2y/h$
N, M, Q	classical stress resultants of beam theory
T_n, V_n, S_n	higher order stress coefficients
U, β, W	classical beam theory deformation variables
U_n, W_n	higher order deformation coefficients
$P_n(\eta)$	Legendre Polynomial of integral order n

INTRODUCTION

ENGINEERING theories for determination of elastic stress and deformation fields in a large class of beam, plate and shell type structures are reasonably well established and have been extensively utilized in design analyses. However, due to approximations inherent in their derivation, these classical theories are limited in their application to configurations having thickness much less than some other characteristic dimension. Thus, these theories cannot be applied with any guarantee of accuracy to the analysis of deep beams, or to thick plate and shell configurations. Also, the classical theories are invalidated when local effects predominate (such as in contact or stress concentration problems); in these instances, the theories fail because the characteristic dimension of the problem is of the same order as the structure thickness.

The range of validity of the classical theories has been extended somewhat by the inclusion of effects such as shear deformation, and transverse normal stress; however, at least in the theory of shells, the inclusion of these effects alone apparently may not necessarily imply greater accuracy in results [1].

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The work to be presented here is an initial effort in a general study to obtain a unified analytical theory suitable for analysis of local effects in beams, plates and shells, as well as for general stress and deformation analysis of deep beams, and of thick plates and shells. The material here only treats the development of a theory suitable for application to deep rectangular strips. This theory is of interest in its own right; it is noted, however, that the investigation is primarily undertaken because it is logically the simplest first step to demonstrate a general philosophy of approach to be later applied to the important problem of thick shell analysis.

No attempt will be made here to completely catalogue previous investigations into this area of interest; however, some brief mention of some of the works which have influenced this author's thinking is in order. Horvay considered end effects in rectangular strips [2] and built up, using an energy method, sets of orthogonal polynomials to represent the self-equilibrating stresses imposed on the end of a rectangular strip. Some of the work here will be a re-examination of Horvay's problem from a different point of view.

Higher order effects due to wave propagation in infinite plates were examined by Mindlin and Medick [3]. Their work was based on systematic reduction of the three dimensional equations of motion to a two dimensional set by expansion of all dependent variables in series involving Legendre Polynomials.

St. Venant effects in thick axisymmetrically loaded cylinders were investigated by Mendelson and Roberts [4], and by Kaehler [5]. Their works had as a basis of approach the solving of all unknown variables exactly in terms of the transverse shear stress. The shear stress distribution was then obtained using collocation methods.

Finally, contact stress effects in thin strips have recently been investigated by Feng and Goodman [6]. Following the lead of Mindlin and Medick, these authors also used Legendre Polynomial representations for the dependent variables. In this author's opinion, however, their final results are seriously in doubt since they predict that the decay rate of the stresses away from the contact region will be a strong function of Poisson's Ratio. This is in direct disagreement with the predictions of the exact theory which requires that the decay functions be governed by a biharmonic equation not involving material properties.

In the following sections, an exact solution to the plane problem of the rectangular strip will be formulated in a manner such that the interrelation between classical beam theory and higher order theories is explicitly brought out. The exact formulation is developed in a form which is particularly amenable to consistent reduction to obtain approximate theories of any order. A possible application of perturbation techniques to the problem is then briefly discussed although not pursued in detail. Finally, a sample solution is obtained using a higher order approximate theory to illustrate local effects.

DERIVATION OF GOVERNING EQUATIONS

Consider the thin rectangular strip of Fig. 1. Assuming generalized plane stress, the governing field equations are:

$$\frac{\partial \sigma_x}{\partial x}+\frac{2}{h}\frac{\partial \sigma_{x\eta}}{\partial \eta}=0 \qquad \eta = 2y/h \qquad \frac{\partial \sigma_{x\eta}}{\partial x}+\frac{2}{h}\frac{\partial \sigma_{\eta}}{\partial \eta}=0$$

$$E\frac{\partial u}{\partial x}=\sigma_x-\nu\sigma_\eta; \qquad \frac{2E}{h}\frac{\partial w}{\partial \eta}=\sigma_\eta-\nu\sigma_x \qquad \frac{2}{h}\frac{\partial u}{\partial \eta}+\frac{\partial w}{\partial x}=\frac{\sigma_{x\eta}}{G}. \tag{1}$$

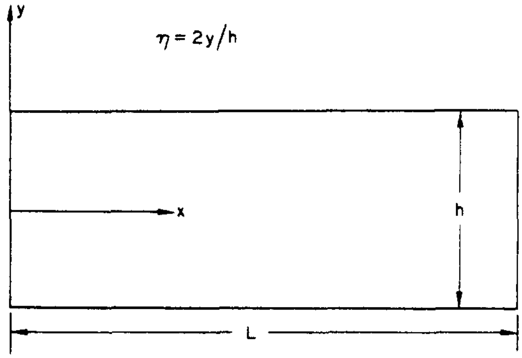

FIG. 1. Beam configuration.

Associated with equations (1) are the boundary conditions:

$$\eta = +1, \text{ all } x: \quad \sigma_{x\eta} = \tau_u(x); \quad \sigma_\eta = q_u(x)$$
$$\eta = -1, \text{ all } x: \quad \sigma_{x\eta} = \tau_L(x); \quad \sigma_\eta = q_L(x) \tag{2}$$

$$x = 0, L, \text{ all } \eta \quad u \text{ or } \sigma_x \text{ specified, and } w \text{ or } \sigma_{x\eta} \text{ specified.} \tag{3}$$

In the work to follow, certain classical definitions are useful:

$$N(x) = \int_{-h/2}^{h/2} \sigma_x \, dy; \qquad M(x) = \int_{-h/2}^{h/2} y\sigma_x \, dy$$
$$Q(x) = \int_{-h/2}^{h/2} \sigma_{xy} \, dy; \qquad \beta(x) = \frac{12}{h^3}\int_{-h/2}^{h/2} yu \, dy \tag{4}$$

In addition, define

$$\tau_u(x)+\tau_L(x) = p(x); \qquad \tau_u(x)-\tau_L(x) = p^*(x)$$
$$q_u(x)+q_L(x) = q(x); \qquad q_u(x)-q_L(x) = q^*(x). \tag{5}$$

Noting boundary conditions (2), assume stresses $\sigma_{x\eta}$ and σ_η in the convergent series representation

$$\sigma_{x\eta} = S_0(x)+S_1(x)\eta+\sum_{n=2}^{\infty} S_n(x)[P_n(\eta)-P_{n-2}(\eta)]$$
$$\sigma_\eta = Z_0(x)+Z_1(x)\eta+\sum_{n=2}^{\infty} Z_n(x)[P_n(\eta)-P_{n-2}(\eta)] \tag{6}$$

where

$$p(x) = 2S_0(x); \qquad p^*(x) = 2S_1(x); \qquad q(x) = 2Z_0(x); \qquad q^*(x) = 2Z_1(x)$$

and

$$S_n(x),\ Z_n(x),\ n \geq 2$$

are as yet undetermined functions. By virtue of equations (4) the function $S_2(x)$ is related to $Q(x)$ in the form:

$$\frac{p(x)}{2} - S_2(x) = Q(x)/h. \tag{7}$$

The dependent variables $u(x,\eta)$, $w(x,\eta)$, $\sigma_x(x,\eta)$ are also expressible in convergent series form:

$$u(x,\eta) = \sum_{n=0}^{\infty} U_n(x)P_n(\eta); \qquad w(x,\eta) = \sum_{n=0}^{\infty} W_n(x)P_n(\eta); \qquad \sigma_x = \sum_{n=0}^{\infty} T_n(x)P_n(\eta). \tag{8}$$

Orthogonality of the Legendre Polynomials makes this representation particularly convenient for expressing boundary conditions at $x = 0, L$. Note that relations

$$U_1(x) = h\beta(x)/2; \qquad T_0(x) = N(x)/h; \qquad T_1(x) = 6M(x)/h^2 \tag{9}$$

exist between classical beam theory variables and the variables of equation (8).

The procedure to obtain higher order theories is now apparent. Having represented all of the dependent variables in the form of convergent series, we substitute into equations (1), and, after eliminating the η dependence, we obtain sets of ordinary differential equations for determination of $U_n(x)$, $W_n(x)$, $T_n(x)$, $S_n(x)$, $Z_n(x)$. The following relations involving Legendre Polynomials and their derivatives are used:

$$P_n(\eta) = (\dot{P}_{n+1} - \dot{P}_{n-1})/2n+1 \qquad n \geq 0 \tag{10}$$

where

$$(\dot{\ }) = \mathrm{d}(\ )/\mathrm{d}\eta; \qquad P_\gamma(\eta) = 0, \gamma < 0$$

$$\dot{P}_n = n(\eta P_n - P_{n-1})/(\eta^2 - 1) = n(n+1)(P_{n+1} - P_{n-1})/(2n+1)(\eta^2 - 1) \tag{11}$$

$$(\eta^2 - 1)P_n = \frac{(n+1)(n+2)}{(2n+1)(2n+3)}(P_{n+2} - P_n) - \frac{n(n-1)}{(2n+1)(2n-1)}(P_n - P_{n-2}) \tag{12}$$

$$(\eta^2 - 1)(P_n - P_{n-2}) = \frac{(n+1)(n+2)}{(2n+1)(2n+3)}(P_{n+2} - P_n) - \frac{2n(n-1)}{(2n+1)(2n-3)}(P_n - P_{n-2}) + \frac{(n-2)(n-3)}{(2n-3)(2n-5)}(P_{n-2} - P_{n-4}). \tag{13}$$

Using equations (6), (8) and (10) in the first of equations (1) yields

$$\sum_{n=0}^{\infty} \left\{ T_n' + \frac{2(2n+1)}{h} S_{n+1} \right\} P_n(\eta) = 0; \qquad (\ )' = \frac{\mathrm{d}(\ )}{\mathrm{d}x}. \tag{14}$$

Since the $P_n(\eta)$ form a complete set of functions in the interval $-1 \leq \eta \leq 1$, to satisfy equations (14) for all x, η we must require that

$$T_n'(x) + \frac{2(2n+1)}{h} S_{n+1}(x) = 0 \qquad n \geq 0. \tag{15}$$

Similarly, substitution into the second of equations (1) yields (after some rearranging of summation indices)

$$\sum_{n=0}^{\infty}\left\{(S'_n - S'_{n+2}) + \frac{2}{h}(2n+1)Z_{n+1}\right\}P_n(\eta) = 0. \tag{16}$$

To ensure satisfaction of the above equation for all η, we must require that

$$Q'(x) + q^*(x) = 0 \tag{17}$$

$$S'_n - S'_{n+2} = -\frac{2(2n+1)}{h}Z_{n+1}(x); \qquad n \geq 1 \tag{18}$$

where we have used equation (7) to obtain equation (17). Since $\sigma_\eta(x, \eta)$ does not appear in boundary conditions at $x = 0$, L, we use equation (18) to eliminate the functions $Z_n(x)$, $n \geq 2$ from further considerations; hence, the normal stress σ_η is written as

$$\sigma_\eta(x, \eta) = Z_0(x) + Z_1(x)\eta - \frac{h}{2}\sum_{n=1}^{\infty}(S'_n - S'_{n+2})\left[\frac{P_{n+1}(\eta) - P_{n-1}(\eta)}{2n+1}\right]. \tag{19}$$

We recall that $Z_0(x)$, $Z_1(x)$, and $S_1(x)$ are known functions dependent solely on surface stresses at $n = \pm 1$.

Substituting (8) and (19) into the third of equations (1) yields an equation similar in form to (14) and (16). Requiring satisfaction of this equation for all η yields the set of ordinary differential equations:

$$\begin{aligned} EU'_0 &= T_0 - \nu Z_0 - \frac{\nu h}{6}(S'_1 - S'_3) \\ EU'_1 &= T_1 - \nu Z_1 - \frac{\nu h}{10}(S'_2 - S'_4) \\ EU'_n &= T_n + \frac{\nu h}{2}\left\{\frac{S'_{n-1}}{2n-1} - \frac{2(2n+1)S'_{n+1}}{(2n-1)(2n+3)} + \frac{S'_{n+3}}{2n+3}\right\} \qquad n \geq 2. \end{aligned} \tag{20}$$

Using (8) in the second stress–strain relation yields a series involving $\dot{P}_n$ and series involving P_n and $P_n - P_{n-2}$. Using equation (11) to eliminate $\dot{P}_n$, multiplying the entire equation by $\eta^2 - 1$, and using equations (12) and (13) eventually leads to the result

$$\begin{aligned} \frac{2E}{h}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2n+3}W_{n+1}(P_{n+2} - P_n) &= \frac{2Z_0}{3}(P_2 - P_0) + \frac{2Z_1}{3}(P_3 - P_1) \\ &+ \sum_{n=0}^{\infty}\nu\frac{(n+1)(n+2)}{2n+3}\left[\frac{T_{n+2}}{2n+5} - \frac{T_n}{2n+1}\right](P_{n+2} - P_n) \\ &- \frac{h}{2}\sum_{n=2}^{\infty}\frac{(n+1)(n+2)}{(2n-1)(2n+1)(2n+3)}[S'_{n-1} - S'_{n+1}](P_{n+2} - P_n) \\ &+ \frac{h}{2}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\left[\frac{2(S'_{n+1} - S'_{n+3})}{2n+1} - \frac{(S'_{n+3} - S'_{n+5})}{2n+7}\right](P_{n+2} - P_n). \end{aligned} \tag{21}$$

The functions $(P_{n+2} - P_n)$ are a complete set for equation (21) in the interval $-1 \leq \eta \leq 1$ and hence to ensure satisfaction of (21), the coefficients of $(P_{n+2} - P_n)$ on each side of

(21) must be equal. Hence, after some manipulation, we obtain:

$$W_1(x) = \frac{Z_0 h}{2E} + \frac{vh}{2E}\left[\frac{T_2}{5} - T_0\right] + \frac{h^2}{4E}\left[\frac{2S_1'}{5} - \frac{3S_3'}{7} + \frac{S_5'}{35}\right] \tag{22}$$

$$W_2(x) = \frac{Z_1 h}{2E} + \frac{vh}{2E}\left[\frac{T_3}{7} - \frac{T_1}{3}\right] + \frac{h^2}{4E}\left[\frac{2S_2'}{21} - \frac{S_4'}{9} + \frac{S_6'}{63}\right] \tag{23}$$

and, for $n \geq 2$

$$\begin{aligned} W_{n+1}(x) = {} & \frac{vh}{2E}\left[\frac{T_{n+2}}{2n+5} - \frac{T_n}{2n+1}\right] \\ & + \frac{h^2}{4E}\left[-\frac{S_{n-1}'}{(2n-1)(2n+1)} + \frac{3S_{n+1}'}{(2n-1)(2n+5)} - \frac{3S_{n+3}'}{(2n+1)(2n+7)} + \frac{S_{n+5}'}{(2n+5)(2n+7)}\right] \end{aligned} \tag{24}$$

Substitution of equations (8) into the final shear stress displacement relation of equations (1) leads to essentially the same type of terms involved in the previous stress–strain relation. After similar manipulations using equations (10), (11) and (12), we obtain the set of ordinary differential equations

$$\frac{2U_{n+1}(x)}{h} = \frac{1}{(2n+1)}\left[\frac{S_n - S_{n-2}}{G} - W_n'\right] - \frac{1}{(2n+5)}\left[\frac{S_{n+2} - S_{n+4}}{G} - W_{n+2}'\right] \qquad n \geq 0. \tag{25}$$

DECOUPLING OF THE EQUATIONS

The set of equations (15), (17), (20), (22), (23), (24), (25) are the governing equations necessary to determine the functions $U_n(x)$, $W_n(x)$, $T_n(x)$, $(n \geq 0)$ and $S_n(x)$ $(n \geq 2)$. It is of interest to note that they have been derived without making use of any orthogonality conditions. These equations can be put in a form which clearly indicates the higher order effects, and which permits consistent truncations to obtain approximate solutions. The procedure is as follows:

1. Equations (22)–(24), and (15) can be used to obtain W_n' for $n \geq 1$ entirely in terms of the stress functions $S_m(x)$.
2. Using these results, the functions $U_n(x)$ $(n \geq 2)$ are obtained solely in terms of the functions $S_m(x)$ from equation (25).
3. Differentiation of equation (20), and application of equations (15) and the results of step 2 yields a set of coupled equations involving only the functions $S_m(x)$. Once these equations are solved all of the other functions are determined in terms of the functions $S_m(x)$.

The equations in their final form are summarized as follows: The equations for the determination of the functions $S(x)$ are:

$$S_{n+1}^{\text{iv}} - \frac{8}{3h^2}(2n-3)(2n+5)S_{n+1}'' + \frac{8}{3h^4}(2n-1)(2n-3)(2n+3)(2n+5)S_{n+1}(x) = F_n(x) \tag{26}$$

where for $n = 2, 3$

$$F_n(x) = \frac{2}{3}\frac{(2n-3)(2n+5)}{(2n+1)(2n+7)}S^{\text{iv}}_{n+3} - \frac{(2n-3)(2n-1)}{6(2n+1)(2n+7)}S^{\text{iv}}_{n+5} - \frac{4}{3h^2}\frac{(2n+3)(2n-1)(2n+5)}{(2n+1)}S''_{n+3}$$
$$+\frac{(2n+3)(2n+5)}{3(2n+1)}\frac{Z'''_{n-2}}{h} + \frac{(2n+5)}{2(2n+1)}S^{\text{iv}}_{n-1} - \frac{4}{3h^2}\frac{(2n+3)(2n+5)(2n-5)}{(2n+1)}S''_{n-1} \quad (27)$$

and for $n \geq 4$

$$F_n(x) = -\frac{(2n+3)(2n+5)}{6(2n+1)(2n+7)}S^{\text{iv}}_{n-3} + \frac{2(2n-3)(2n+5)}{3(2n+1)(2n-5)}S^{\text{iv}}_{n-1} + \frac{2(2n-3)(2n+5)}{3(2n+1)(2n+7)}S^{\text{iv}}_{n+3}$$
$$-\frac{(2n-3)(2n-1)}{6(2n+1)(2n+7)}S^{\text{iv}}_{n+5} - \frac{4(2n-3)(2n+3)(2n+5)}{3(2n+1)h^2}S''_{n-1}$$
$$-\frac{4(2n-3)(2n-1)(2n+5)}{3(2n+1)h^2}S''_{n+3}. \quad (28)$$

Using equations (4), the functions $U_n(x)$ are given as:

$n = 2, 3$

$$U_n(x) = -\frac{h^3}{8E}\left[\frac{3S''_{n-1}}{(2n-3)(2n-1)(2n+3)} - \frac{6(2n+1)S''_{n+1}}{(2n-1)(2n-3)(2n+3)(2n+5)}\right.$$
$$\left.+\frac{4S''_{n+3}}{(2n-1)(2n+3)(2n+7)} - \frac{S''_{n+5}}{(2n+3)(2n+5)(2n+7)}\right] \quad (29)$$
$$-\frac{h^2 Z'_{n-2}}{4E(2n-1)(2n-3)} + \frac{h(1+\nu/2)}{E}\left[\frac{S_{n-1}}{2n-1} - \frac{2(2n+1)S_{n+1}}{(2n-1)(2n+3)} + \frac{S_{n+3}}{2n+3}\right]$$

$n \geq 4$

$$U_n(x) = -\frac{h^3}{8E}\left[\frac{-S''_{n-3}}{(2n-3)(2n-1)(2n-5)} + \frac{4S''_{n-1}}{(2n-5)(2n-1)(2n+3)}\right.$$
$$-\frac{6(2n+1)S''_{n+1}}{(2n-1)(2n-3)(2n+3)(2n+5)} + \frac{4S''_{n+3}}{(2n-1)(2n+3)(2n+7)} \quad (30)$$
$$\left.-\frac{S''_{n+5}}{(2n+3)(2n+5)(2n+7)}\right] + \frac{h(1+\nu/2)}{E}\left[\frac{S_{n-1}}{2n-1} - \frac{2(2n+1)S_{n+1}}{(2n-1)(2n+3)} + \frac{S_{n+3}}{2n+3}\right].$$

The normal stress functions $T_n(x)$ for $n \geq 2$ are given by the last of equations (20), while the functions $W_n(x)$ ($n \geq 1$) are given by equations (22)–(24). The above system constitutes a set of $4n-8$ equations. The assumed series solutions contain $4n-2$ unknown functions of x. The remaining six equations are obtained from previous results which have so far not been utilized to obtain equations (26)–(30); namely equations (15) for $n = 0, 1$, equation (17), the first two of (20), and equation (25) for $n = 0$. These six equations can be written in terms of classical beam theory resultants (using equations (7) and (9)) as:

$$N'+p^*(x) = 0; \qquad EU'_0 = N/h - \nu\left[\frac{q(x)}{2}+\frac{h}{6}\left(\frac{p^{*\prime}}{2}-S'_3\right)\right]$$

$$M'-Q+\frac{hp(x)}{2} = 0; \qquad \frac{Eh\beta'}{2} = \frac{6M}{h^2}-\nu\left[\frac{q^*(x)}{2}+\frac{h}{10}\left(\frac{p'}{2}-\frac{Q'}{h}-S'_4\right)\right] \tag{31}$$

$$Q'+q^*(x) = 0; \qquad \beta+W'_0 = \frac{6Q}{5Gh}-\frac{p(x)}{10G}+\frac{S_4(x)}{5G}+\frac{W'_2}{5}.$$

To complete the problem formulation, we must specify the boundary conditions in terms of the series coefficients. Using the orthogonality relations for Legendre Polynomials, we have at $x = 0, L$

$$u = \sum_{n=0}^{\infty} \bar{u}_n P_n(\eta); \qquad \sigma_x = \sum_{n=0}^{\infty} \bar{\sigma}_{x_n} P_n(\eta)$$

$$w = \sum_{n=0}^{\infty} \bar{w}_n P_n(\eta); \qquad \sigma_{x\eta} = \sum_{n=0}^{\infty} \bar{\sigma}_{x\eta_n} P_n(\eta) \tag{32}$$

where

$$(2n+1)\bar{u}_n = 2\int_{-1}^{1} uP_n(\eta)\,d\eta$$

$$(2n+1)\bar{\sigma}_{x\eta_n} = 2\int_{-1}^{1} \sigma_{x\eta}P_n(\eta)\,d\eta \text{ etc.} \tag{33}$$

Hence, the required boundary conditions at $x = \bar{x}$ are

$$\bar{u}_n = U_n(\bar{x}) \quad \text{or} \quad \bar{\sigma}_{x_n} = T_n(\bar{x})$$

and

$$\bar{w}_n = W_n(\bar{x}) \quad \text{or} \quad \bar{\sigma}_{x\eta_n} = (S_n(\bar{x})-S_{n+2}(\bar{x})). \tag{34}$$

Having completed the development of the governing equations and boundary conditions, we comment briefly on the results. The expansion of all of the dependent variables in series of Legendre Polynomials has enabled us to decouple the boundary conditions, but it has led also to a formidable and infinite array of coupled ordinary differential equations. A closed form exact solution to these equations is not obtainable: the best that can be expected is a closed form approximate solution obtained by some *consistent* truncation of each series. In the author's opinion, the advantage of the formulation developed here is twofold: first, the development in terms of Legendre Polynomials assures convergence of the series, so that the consistent inclusion of more terms in each series to obtain higher approximations leads to a better solution; secondly, the development presented here seems to indicate quite clearly the truncation necessary to obtain a consistent theory of any order. For example, suppose it is desired to develop a consistent approximate theory in which the highest order term in the shear stress involves $S_{n+2}(x)$. The second of equations (34) immediately implies that for consistency in the boundary conditions, the series for $w(x, \eta)$ must truncate after the term involving $W_n(x)$. Similarly, equations (15) imply that the series solution for $\sigma_x(x, \eta)$ should include the term involving $T_{n+1}(x)$. Finally, (noting the order of the normal stress coefficients) a consistent boundary condition, as given by the first of equations (34), requires that the series for $u(x, \eta)$ should be truncated after the term involving $U_{n+1}(x)$. Hence, we may label as an nth order approximation a theory which contains all terms in the respective series up to and including the following terms:

$$n\text{th order approx. } (S_{n+2}(x), W_n(x), T_{n+1}(x), U_{n+1}(x)). \tag{35}$$

The single exception to equation (35), of course, will be for $n = 0$ where we should also include the functions $T_0(x)$ and $U_0(x)$. For example, we give the governing equations and boundary conditions for a consistent second order theory which includes the first higher order effect in both the symmetric and the anti-symmetric distributions.

Symmetric problem

$$\left.\begin{aligned} N' + p^*(x) &= 0 \\ EhU_0' &= N - \frac{\nu h}{2}\left[q(x) + \frac{p^{*\prime}h}{6}\right] + \frac{\nu h^2}{6}S_3'(x) \end{aligned}\right\} \tag{36}$$

$$S_3^{\text{iv}} - \frac{24}{h^2}S_3'' + \frac{504}{h^4}S_3 = \frac{21}{10}\frac{q'''(x)}{h} + \frac{9}{20}p^{*\text{iv}} + \frac{42}{5h^2}p^{*\prime\prime} \tag{37}$$

$$U_2(x) = -\frac{h^3}{8E}\left[\frac{p^{*\prime\prime}}{14} - \frac{10}{63}S_3''\right] - \frac{q'(x)h^2}{24E} + \frac{h(1+\nu/2)}{E}\left[\frac{p^*(x)}{6} - \frac{10}{21}S_3(x)\right] \tag{38}$$

$$T_2(x) = EU_2' - \frac{\nu h}{2}\left[\frac{p^{*\prime}}{6} - \frac{10}{21}S_3'\right] \tag{39}$$

$$W_1(x) = \frac{q(x)h}{4E} + \frac{\nu h}{2E}\left[\frac{T_2}{5} - \frac{N}{h}\right] + \frac{h^2}{4E}\left[\frac{p^{*\prime}}{5} - \frac{3}{7}S_3'\right]. \tag{40}$$

At $x = 0, L$, we have

$$U_0 \text{ or } N \text{ specified}: U_2 \text{ or } T_2 \text{ specified, and } W_1 \text{ or } S_3 - p^*/2 \text{ specified.} \tag{41}$$

Anti-symmetric problem

$$\left.\begin{aligned} Q' + q^*(x) &= 0 \\ M' - Q + \frac{h}{2}p(x) &= 0 \\ \frac{Eh^3}{12}\beta' &= M - \frac{\nu h^2}{2}\left[\frac{3q^*}{5} + \frac{hp'}{20}\right] + \frac{\nu h^3}{60}S_4' \\ \beta + W_0' &= \frac{6Q}{5Gh} - \frac{p(x)}{10G} + \frac{S_4}{5G} + \frac{W_2'}{5} \end{aligned}\right\} \tag{42}$$

$$S_4^{\text{iv}} - \frac{88}{h^2}S_4'' + \frac{3960}{h^4}S_4 = \frac{33}{14h}q^{*\prime\prime\prime} + \frac{11}{14}\left[\frac{p^{\text{iv}}}{2} - \frac{Q^{\text{iv}}}{h}\right] - \frac{132}{7h^2}\left[\frac{p''}{2} - \frac{Q''}{h}\right] \tag{43}$$

$$U_3(x) = \frac{-h^3}{8E}\left[\frac{1}{45}\left(\frac{p''}{2}-\frac{Q''}{h}\right)-\frac{14}{495}S_4''\right]-\frac{h^2q^{*\prime}}{120E}+\frac{h(1+\nu/2)}{E}\left[\frac{1}{5}\left(\frac{p}{2}-\frac{Q}{h}\right)-\frac{14}{45}S_4\right] \tag{44}$$

$$T_2(x) = EU_3'-\frac{\nu h}{2}\left[\frac{1}{5}\left(\frac{p'}{2}-\frac{Q'}{h}\right)-\frac{14}{45}S_4'\right] \tag{45}$$

$$W_2(x) = \frac{q^*h}{12E}+\frac{\nu h}{2E}\left[\frac{T_3}{7}-\frac{2M}{h^2}\right]+\frac{h^2}{4E}\left[\frac{2}{21}\left(\frac{p'}{2}-\frac{Q'}{h}\right)-\frac{S_4'}{9}\right]. \tag{46}$$

At $x = 0, L$

$$\left.\begin{array}{l} W_0 \text{ or } Q/h \text{ specified, and } h\beta/2 \text{ or } 6M/h^2 \text{ specified} \\ W_2 \text{ or } \left(\frac{p}{2}-\frac{Q}{h}-S_4\right) \text{ specified and } U_3 \text{ or } T_3 \text{ specified} \end{array}\right\}. \tag{47}$$

Note from equations (37)–(39) and (43)–(45) that the equations to determine the higher order stresses are independent of material properties; as noted in the introduction, a theory not possessing this characteristic cannot hope to give valid approximate solutions for the field variables of the rectangular strip problem.

Solution of the above equations, subject to the given boundary conditions, yields stresses and displacements as

$$\begin{aligned} u(x,\eta) &= U_0(x)+\frac{h\beta(x)}{2}\eta+U_2(x)P_2(\eta)+U_3(x)P_3(\eta) \\ \sigma_x(x,\eta) &= \frac{N(x)}{h}+\frac{6M(x)}{h^2}\eta+T_2(x)P_2(\eta)+T_3(x)P_3(\eta) \\ w(x,\eta) &= W_0(x)+W_1(x)\eta+W_2(x)P_2(\eta) \\ \sigma_{x\eta}(x,\eta) &= \frac{p(x)}{2}+\frac{p^*(x)}{2}\eta+\left[\frac{p(x)}{2}-\frac{Q(x)}{h}\right](P_2(\eta)-1)+S_3(x)(P_3(\eta)-\eta) \\ &\quad +S_4(x)(P_4(\eta)-P_2(\eta)) \\ \sigma_\eta(x,\eta) &= \frac{q(x)}{2}+\frac{q^*(x)}{2}\eta-\frac{h}{6}\left[\frac{p^{*\prime}}{2}-S_3'(x)\right](P_2(\eta)-1) \\ &\quad -\frac{h}{10}\left[\frac{p'}{2}+\frac{q^*}{h}-S_4'(x)\right](P_3(\eta)-\eta). \end{aligned} \tag{48}$$

Classical beam theory for bending and extension of the strip (the $n = 0$ approximation) is seen to be just equations (36) and (42) with $S_3(x)$, $S_4(x)$, and $W_2(x)$ omitted. Note that the lowest order theory includes shear deformation and transverse normal stress effects. Further reduction to a Kirchhoff type theory neglecting such effects can only be justified numerically, or by order of magnitude analysis of the zeroth approximation equations. This is not surprising as we have insisted that a minimum criterion for an acceptable theory be that *all* of the equations of elasticity are approximately satisfied for any truncation of the series solutions.

The exact solution to the rectangular strip problem has been essentially reduced to a determination of the functions $S_n(x)$, $n \geq 3$ by solving the coupled equations (26)–(28).

We examine the consequences of determining the homogeneous solutions $S_n(x)$ by neglecting all coupling terms in the equations: that is, we assume $S_n(x)$ is given by the equation

$$S^{\mathrm{iv}}_{n+1}(x)-\frac{8}{3h^2}(2n-3)(2n+5)S''_{n+1}+\frac{8}{3h^4}(2n-1)(2n-3)(2n+3)(2n+5)$$
$$\times S_{n+1}(x)\approx 0 \qquad n\geq 2. \tag{49}$$

The solution is given as

$$S_{n+1}(x)=\sum_{i=1}^{4}A_{i_{(n+1)}}\exp\left(\mu^{(i)}_{n+1}\frac{x}{h}\right)$$

with the characteristic roots given as

$$\mu_{n+1} = \pm 2[(2n-3)(2n-1)(2n+3)(2n+5)/6]^{1/4}\exp(\pm i\phi_{n+1}/2);$$
$$\phi_{n+1} = \tan^{-1}\left[1{\cdot}5\frac{(2n-1)(2n+3)}{(2n-3)(2n+5)}-1\right]^{1/2} \qquad n\geq 2. \tag{50}$$

The two lowest root sets are

$$\mu_3 = \pm 4{\cdot}155\pm 2{\cdot}290\,i; \qquad \mu_4 = \pm 7{\cdot}300\pm 3{\cdot}120\,i \tag{51}$$

These values are in good agreement with those obtained by Horvay using an approximate energy approach [2]. However, as n increases, we obtain from equation (50)

$$\mu_5 = \pm 10{\cdot}05\pm 3{\cdot}80\,i; \qquad \mu_6 = \pm 12{\cdot}90\pm 3{\cdot}99;\ \text{etc.} \tag{52}$$

and the agreement with Horvay's approximate results becomes somewhat poorer. The above computations and comparison with other works indicates that coupling effects in equations (26)–(28) become more important for large values of n, and are negligible for the lowest values of n. An area for further study immediately suggests itself. It may be possible to construct a perturbation scheme of solution to approximately solve equations (26)–(28) for all n. Such a scheme would introduce a perturbation parameter ε such that the right hand side of equation (26) would be $\varepsilon F_n(x)$ instead of $F_n(x)$. The perturbation scheme would evolve a solution for $S_{n+1}(x)$ in powers of ε as well as a solution for μ_{n+1} in powers of ε. Setting $\varepsilon = 1$ in the result may yield improved approximate solutions. A perturbation technique well suited for this suggested approach has been recently investigated by the author [7, 8]; its application to the rectangular strip problem in the above suggested manner is currently being investigated and will be reported on at a later date.

APPLICATION OF THE THEORY

As an example of the possible utilization of the approximate theory contained in equations (36)–(46), we examine the beam problem illustrated in Fig. 2. It is of interest to determine the stress distribution in the immediate vicinity of the highly localized pressure distribution. Specializing equations (36)–(46) to the particular pressure distribution of Fig. 2 yields the following governing equations which are of interest in determining local stresses:

$$\left.\begin{aligned} N &= 0 \\ S_3^{iv} - \frac{24}{h^2}S_3'' + \frac{504}{h^4}S_3 &= -\frac{21}{20}\frac{P}{h}\left(\frac{a}{h}\right)^4 \exp(-ax/h) \\ U_2(x) &= \frac{5h^3}{252}S_3''' - \frac{10h}{21E}(1+\nu/2)S_3(x) + \frac{Pa^2}{48E}\exp(-ax/h) \\ T_2(x) &= \frac{5h^3}{252}S_3''' - \frac{10}{21}hS_3' - \frac{Pa^3}{48h}\exp(-ax/h) \end{aligned}\right\} \tag{53}$$

$$\left.\begin{aligned} Q(x) &= \frac{P}{2}[\exp(-ax/h)-1] \\ M(x) &= \frac{PL}{4}\left[1-\frac{2x}{L}-\frac{2h}{aL}\exp(-ax/h)\right] \\ S_4^{iv} - \frac{88}{h^2}S_4'' + \frac{3960}{h^4}S_4 &= -\frac{11P}{7h}\left(\frac{a}{h}\right)^4(1-6/a^2)\exp(-ax/h) \\ U_3(x) &= \frac{7h^3}{1980E}S_4'' - \frac{14h}{45E}(1+\nu/2)S_4 + \frac{P}{10E}\left[\frac{a^2}{18}-(1+\nu/2)\right]\exp(-ax/h) + \frac{P(1+\nu/2)}{10E} \\ T_3(x) &= \frac{7h^3}{1980}S_4'' - \frac{14h}{45}S_4' - \frac{Pa}{10h}\left[\frac{a^2}{18}-1\right]\exp(-ax/h) \end{aligned}\right\}. \tag{54}$$

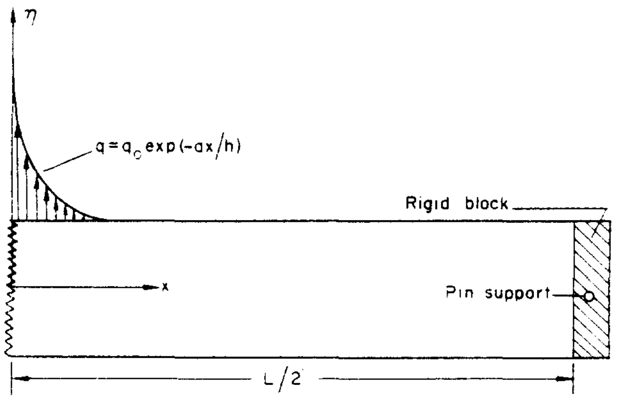

FIG. 2. Beam loading for sample problem.

In obtaining the above equations, we have already made use of the boundary conditions

$$N(L/2) = M(L/2) = Q(0) = 0 \tag{55}$$

Solutions for the shear stress functions $S_3(x)$ and $S_4(x)$ are

$$S_3(x) = C_0EC[x/h]+C_1ES[x/h]+C_2EC\left[\frac{L}{2h}-\frac{x}{h}\right]+C_3ES\left[\frac{L}{2h}-\frac{x}{h}\right]$$
$$-\frac{21P}{20\gamma h}\exp(-ax/h) \tag{56}$$

$$S_4(x) = D_0\overline{EC}[x/h]+D_1\overline{ES}[x/h]+D_2\overline{EC}\left[\frac{L}{2h}-\frac{x}{h}\right]+D_3\overline{ES}\left[\frac{L}{2h}-\frac{x}{h}\right]$$
$$-\frac{11\mu P}{7h}\exp(-ax/h) \tag{57}$$

where

$$\gamma = 1-24/a^2+504/a^4; \qquad \mu = (1-6/a^2)/(1-88/a^2+3960/a^4) \tag{58}$$

and

$$\begin{aligned} EC(y) &= \exp(-4{\cdot}155y)\cos(2{\cdot}29y); & ES(y) &= \exp(-4{\cdot}155)\sin(2{\cdot}29y) \\ \overline{EC}(y) &= \exp(-7{\cdot}300y)\cos(3{\cdot}12y); & \overline{ES}(y) &= \exp(-7{\cdot}300y)\sin(3{\cdot}12y). \end{aligned} \tag{59}$$

To obtain the stress distribution near $x = 0$ neglect end interactions so that only constants C_0, C_1, D_0, D_1 need be evaluated. Equations for determination of these constants are obtained from the condition that at $x = 0$,

$$S_3(0) = S_4(0) = U_2(0) = U_3(0) = 0. \tag{60}$$

Once the integration constants are evaluated, the local stress distributions are obtained from equations (48). In Fig. 3, for example, are plotted the local shear and axial stress distributions for the case of the loading decay parameter "a" equal to 20. This value for "a" gives a reasonable representation of a localized loading; the applied pressure is reduced to $1/e^2$ of its value at $x = 0$ at the point $x = 0{\cdot}1\,h$.

In Fig. 3(a) is shown the shear stress distribution as a function of η at various stations along the beam. As is to be expected, the distribution in the immediate vicinity of the loading exhibits considerable deviation from the classical assumed parabolic distribution.

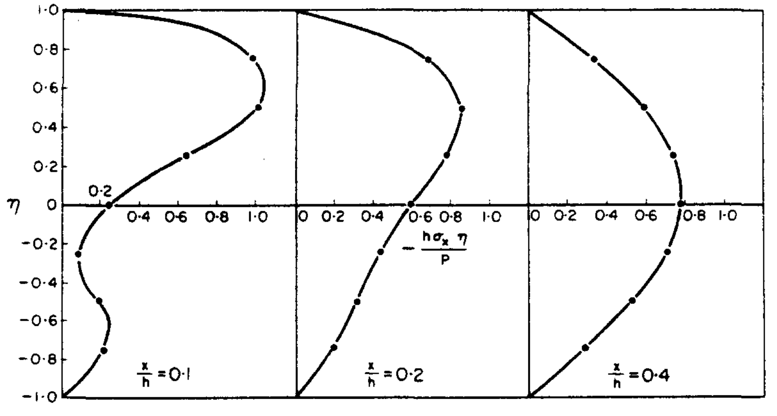

FIG. 3(a). Shear stress profile near local loading.

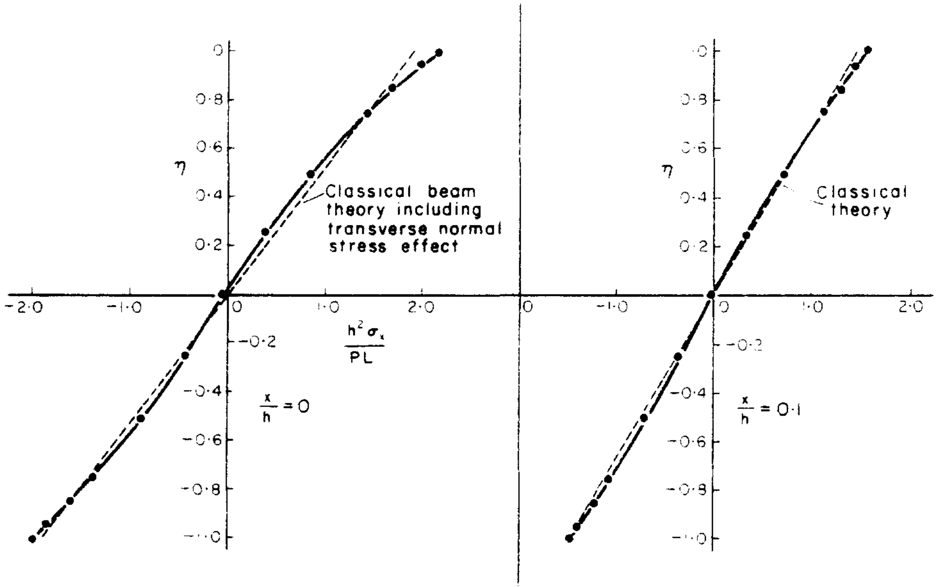

FIG. 3(b). Axial stress profile ($h/L = 0{\cdot}083$).

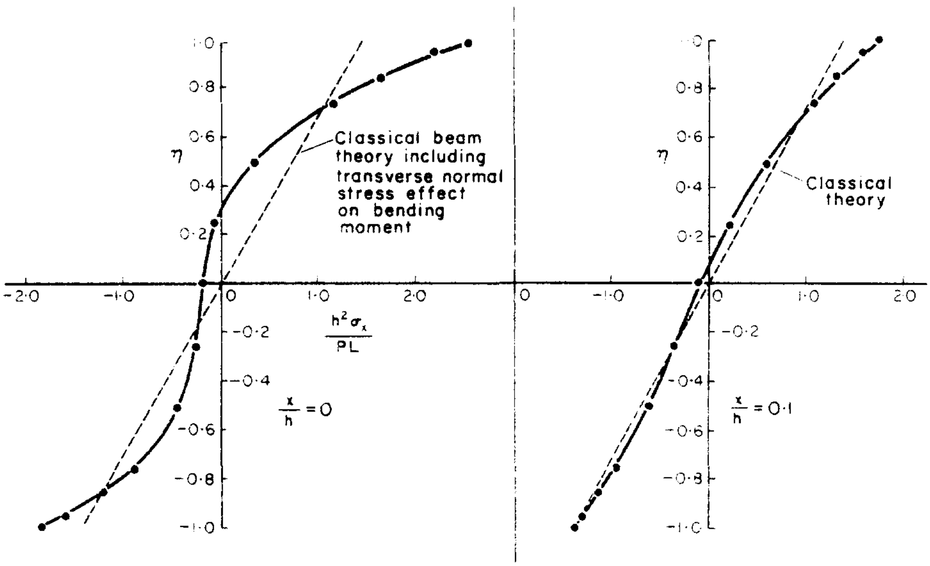

FIG. 3(c). Axial stress profile ($h/L = 0{\cdot}35$).

As x is increased, the transformation to a parabolic distribution becomes evident. For all practical purposes, the parabolic distribution is essentially established at $x = 0{\cdot}4h$. Note that the shear stress distribution is independent of the thickness to length ratio of the beam.

In Figs. 3(b), 3(c) and 3(d) are plotted similar results for the axial stress distribution. The solution for the axial stress is a function of the thickness to length ratio h/L. The

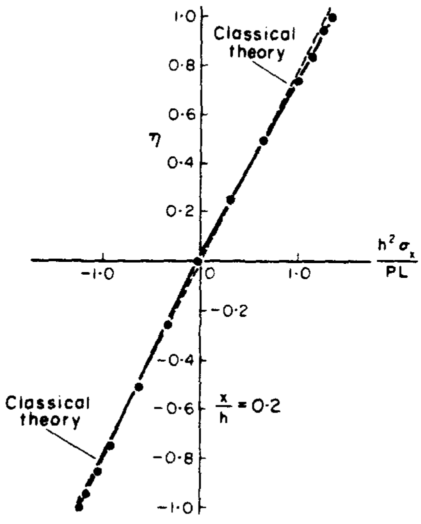

FIG. 3(d). Axial stress profile (h/L = 0·35).

first ratio gives a beam configuration that is usually considered as being describable by a classical beam theory. The second ratio gives a beam configuration that can be classed as moderately thick; beams having this depth to length ratio have heretofore been examined using some theory including transverse normal stress and shear deformation. (Such theories are classified in the terminology of this paper as zeroth order theories.) Figure 3(b) shows that for the thinner beam, local effects are not too important (for the particular loading used here) and the stress distribution is adequately described for design purposes by the linear distribution of classical theory. It should be noted, however, that there is a 12 per cent error in the prediction of maximum stress at $x = 0$; hence, in certain design situations, the application of a higher order theory might be in order even for the thin beam described. Note that at $x/h = 0{\cdot}1$, the linear distribution has essentially been established.

For the thicker beam ($h/L = 0{\cdot}35$), the situation is considerably different. A considerable deviation from linearity exists in the vicinity of the local loading and gives rise to maximum stresses which are much higher than the linear distribution prediction. If h/L were further increased, the deviations would of course be more and more significant and eventually would be of a non-negligible character over the entire length of the beam. From the work done so far, it has not been possible to establish an upper limit to the application of the second order theory used in these calculations, but it is quite likely that for much larger h/L ratios, a higher order theory would be in order. Further study of this question is now going on and will be reported on at a later date.

For design purposes, an important result is the value of the maximum axial stress. The table below presents an indication of the per cent error involved when one uses a classical theory for stress prediction. The numbers in the table are values of the error parameter E defined as

$$E = \frac{\sigma_{x2} - \sigma_{x0}}{\sigma_{x2}} \times 100 \tag{61}$$

where

$$\sigma_{x2} = \text{axial stress predicted by second order theory}$$

$$\sigma_{x0} = \text{axial stress predicted by classical beam theory}$$

TABLE 1. VALUES OF E FOR DIFFERENT h/L RATIOS AT DIFFERENT AXIAL STATIONS

h/L \ x/h	0	0·1	0·2
0·083	$E = 12{\cdot}15$	5·45	0·54
0·35	$E = 43{\cdot}37$	20·57	2·49

CONCLUSIONS

This work presents a derivation of a beam theory incorporating higher order St. Venant effects. The governing equations are derived by systematic reduction of the two dimensional elasticity equations. Although the reduction initially yields an infinite set of coupled ordinary differential equations which describe the behavior of the field variables, a consistent truncation scheme is easily evolved to reduce the set to a finite array incorporating as many higher order effects as deemed necessary for particular applications. The equations for a second order theory involving the first two higher order effects are presented and a sample calculation is included to show effects of local loading.

It is felt that the second order theory presented here can be effectively used to study contact problems involving rectangular strips. Although application to beam problems is of considerable interest; of far greater interest, in the opinion of the author, is the possibility of extending the derivation to shell configurations to obtain consistent theories exhibiting local effects. (Such a theory when evolved should also shed some light on the problem of determining a best first order shell theory.) A shell theory exhibiting such local effects may match almost exactly all of the 3-D elasticity results presented in [1], (which were heretofore used to examine and compare only first order shell theories).

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Абстракт—Исследуются регулированые уравнения прямоугольной полосы с целью получения решений, которые указывают на зависимость между классической теорией балки и теориями высших рангов. Строгие уравнения плоской упругости сводятся к сопряженной системе обыкновенных дифференцияльных уравнений путем представления всех зависимых переменных в виде решений в рядах, которые заключают полиномы Лежендра в координате толщины. Полиномы Лежандра особенно полезные для этого метода в виду их удобной формулировки свойств полноты, сходимости и ортогональности, и поэтому, что обыкновенные результаты напряжений классической теории балок оказываются по натуре коэффициентами полиномов P_1 и P_2. Представленные в таком виде сопряженные обыкновенные дифференциальные уравнения позваляют на то, что можно отбросить часть рядов с целью получения немедленно приближенного решения. Исследуются эффекты сопряжения и предполагается метод определения приближенного решения для полных сопряженных уравнений, не отбрасывая части рядов в решениях в полиномах. Разработано детально пробную задачу с целью иллюстрации применения новой приближенной теории.